Sand castle

D. Oliinychenko

Wolfram, MIPT, Russia

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Problem statement

- Estimate the strength of wet sand.
- Given base area find maximal height.

Guinness book

The highest castle:

- 11.53 м
- Connecticute, USA
- 800 tons of sand
- 1400 voluteers
- 50 days

J. B. Morgan

General considerations

- Problem parameters
  - a[m] - sand size
  - R[m] - base radius (or base $\sqrt{S}$)
  - H[m] - castle height
  - σ[N/m] - water surface tension
  - $\rho[kg/m^3]$ - sand density
  - $E_s[Pa = N/m^2]$ - dry sand Young modulus
  - ϕ - sand humidity
  - g$[m/s^2]$

General considerations

- Dimensionsless parameter combinations and assumptions
  - $\phi \ll 100\%$ - sand humidity
  - $a/R \ll 1$ - homogenious medium
  - $R/H \ll 1$ - rod case
  - $\frac{\sigma}{\rho g H a} = \frac{\sigma a^2 \cdot m/\rho a^3}{mgh}$ - water surface energy/potential energy
  - $\frac{\sigma}{E_s a}$
- Estimate: D. Quere, Non-sticking drops. Rep. Prog. Phys. 68, 2005.
  - $\frac{\sigma}{\rho g H a} \sim 1$
  - Height 10-20 cm (!)
  - Does not depend on area (!)
  - Criticized.

Section 2

How column sticks together: microlevel

Water bridges

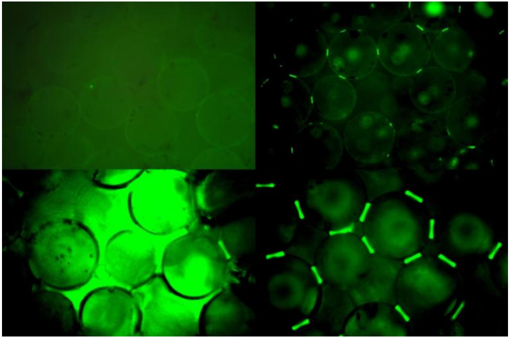
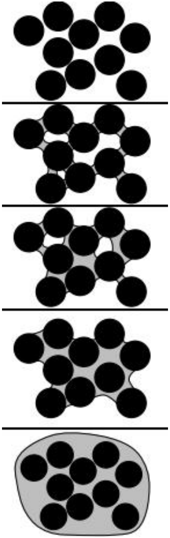


Figure: Taken from [7], fluorescent microscopy photo. Glass+water. Volume water percent (clockwise from left up - 0.1%, 0.3%, 3%, 6%)

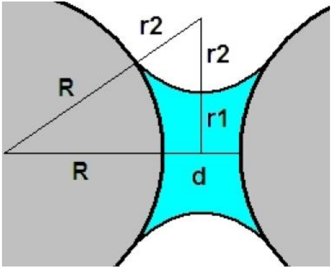


Water bridges: sticking force

$$(R + r_2)^2 = (R + d/2)^2 + (r_1 + r_2)^2 \quad (1)$$

$$F = 2\pi r_1 \sigma - \pi r_1^2 \Delta p = \pi r_1 \sigma + \pi \sigma \frac{r_1^2}{r_2} \quad (2)$$

$$d = 0 \to F = 2\pi\sigma R \quad (3)$$



- For a small amount of water is necessary to consider irregularities of spheres
- For large amount of water F=0
- Dependence on humidity: $F = 2\pi\sigma R \cdot f(\phi)$

$$\Delta p_{Lapl} = \sigma \left[\frac{1}{r_1} - \frac{1}{r_2} \right] \quad (4)$$

Section 3

Column destrction: 2 mechanisms

Shift destruction

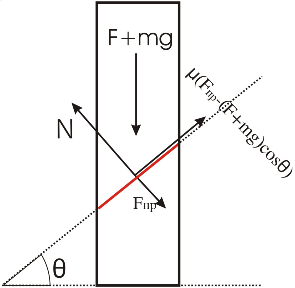
Shift condition

$$(F + mg)(sin\theta - \mu cos\theta) - \frac{2}{\pi}\mu F_{np} > \frac{2}{\pi}F_{np}$$

$$F_{np} = \frac{2\pi a\sigma}{(2a)^2} \cdot \frac{S}{cos\theta}$$

Multipliers $\frac{2}{\pi}$ - from angle averaging.

$$(F + mg)(sin2\theta - \mu(cos2\theta + 1)) > (\mu + 1)\frac{2\sigma}{a}S$$

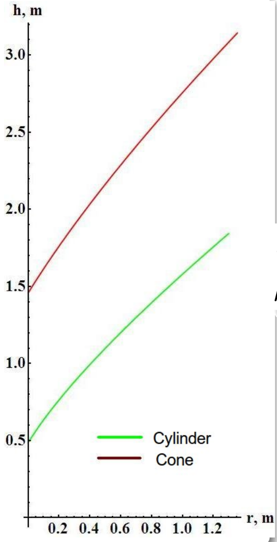


Destruction angle: maximal tangential tension

$$tg2\theta_m = -1/\mu,\ r \ll h$$

- F_{np} - sticking force
- μ - internal friction coeff.
- F - external force

Destruction from shift: cylinder and cone



$\frac{\sigma}{\rho g a} = 10cm$

$\mu = 0.5$

Cylinder: critical height

$$\sqrt{\mu^2+1}\sqrt{h_{cr}^2+r^2} - \mu h - r = 2(\mu+1)\frac{\sigma}{\rho g a}$$

$h_{cr} = \frac{\sigma}{\rho g a}\frac{2(\mu+1)}{\sqrt{\mu^2+1}-\mu}$ - at $r \ll h$

Cone: critical height

$h_{cr} = \frac{\sigma}{\rho g a}\frac{2(\mu+1)}{\sqrt{\mu^2+1}}\frac{3 sin\alpha}{1-cos(\alpha-\theta_0)}$ at $\alpha > \theta_0$

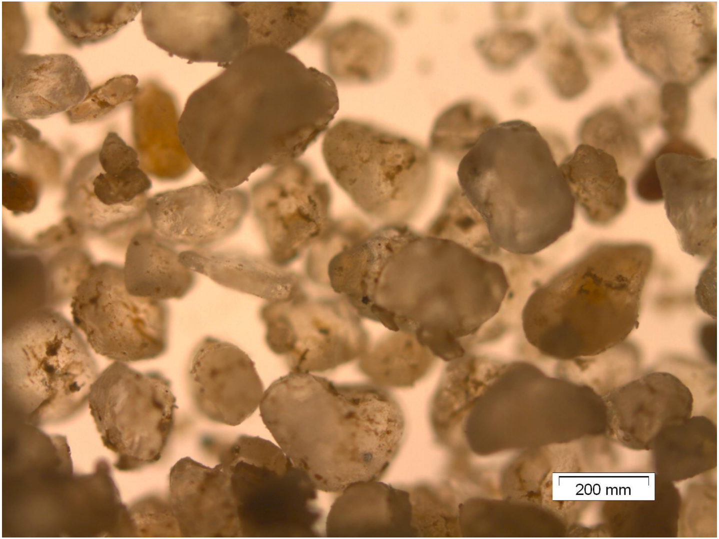
Cone will never destroy at $\alpha < \theta_0$

α - angle at base

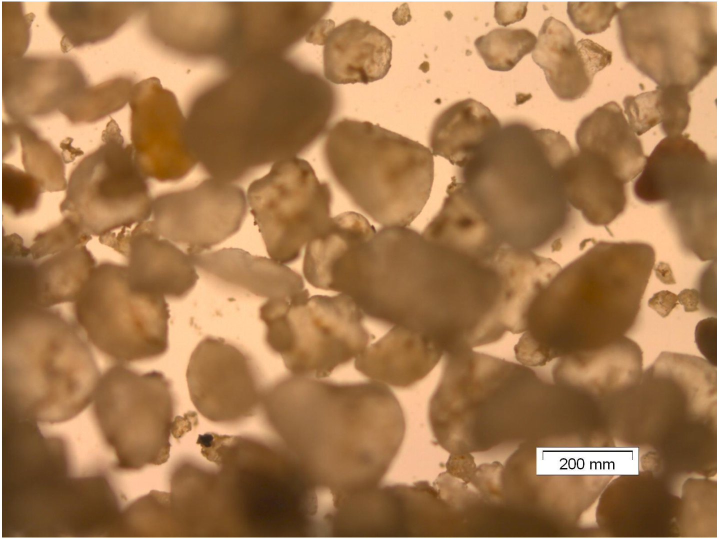
$tg\theta_0 = \mu$

Measurements for sand from Wolga river: μ

Measurements for sand from Wolga: a



Measurements for sand from Wolga: a



Measurements for sand from Wolga: a

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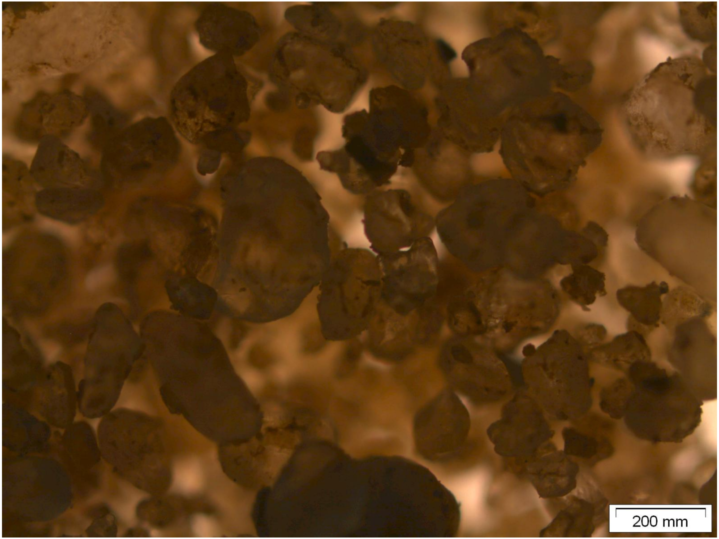


Measurements for sand from Wolga

$$\mu_{max} = 0.75$$

$$a = 20 - 200\,\mu m$$

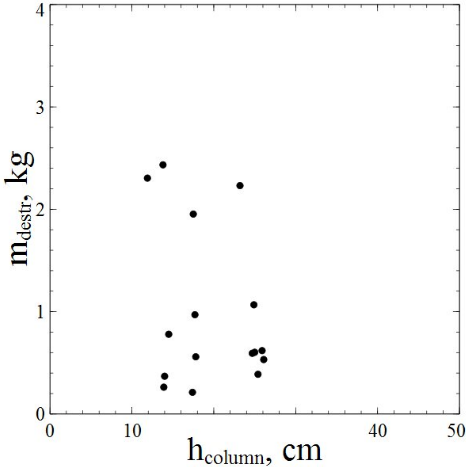
$$\rho = 1430 kg/m^3$$

$$\frac{\sigma}{\rho g a} = \frac{72 mN/m}{1430 kg/m^3 \cdot 10 m/s^2 \cdot 0.1 mm} = 2 - 25\ cm$$

Tamper effect: smaller grains play role.

On strength and tampering

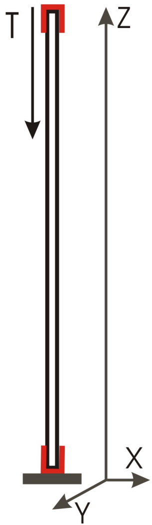
Mass of weight, destroying column on column height



Buckling instability: simplest example

Light rod of constant round cross-section with fixed ends is pressed with force T. For arbitrary cross-section:

- $\overrightarrow{dM} + [\overrightarrow{dl} x \overrightarrow{F}] = 0$
- $F_z = -T$
- $M_x = -EIY'', M_y = -EIX'', M_z = 0$
- $EIX''' + TX' = 0$, $X(0) = X(H) = X'(0) = X'(H) = 0$
- $X' = U, U'' + \frac{T}{EI}U = 0, U(0) = U(H) = 0$
- $U = A \cdot sin\sqrt{T/EI}z$
- $\sqrt{T/EI}z = \pi n$
- $T_{cr} = \frac{EI}{H^2}\pi^2$

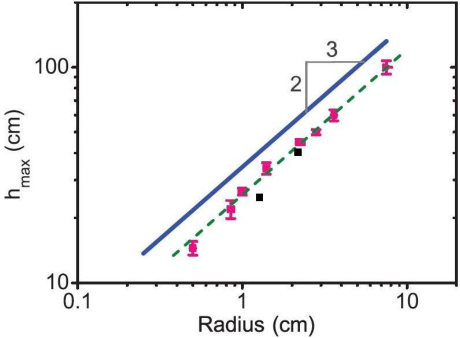


Buckling of cylindrical column under its own weight

Buckling of cylindrical column under its own weight

Landau, Lifshitz b.7:

$H_{max} = 1.98 \left(\frac{EI}{\rho g}\right)^{1/3}, \ I = R^2/4$

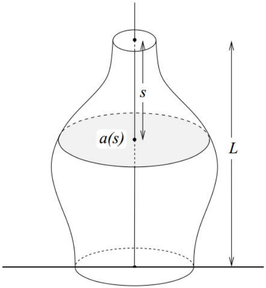


$H_{max}(R)$ for cylindrical column of wet sand [6]

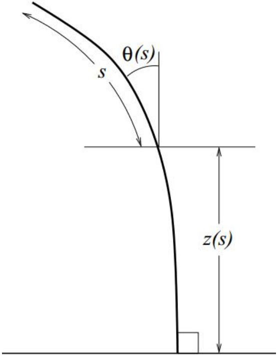
$H_{max} \sim R^{0.7 \pm 0.05}$

Buckling of arbitrary shape column under its own weight[5]

$$e[\theta] = \int_0^L \frac{1}{2} I(s)\theta_s^2(s)ds + \rho g \int_0^L a(s)z(s)ds$$



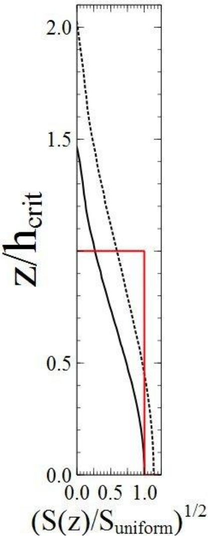
- $I(s) = cEa^2(s)$
- Fix the volume: $V = \int_0^L a(s)ds$
- Dimensionless: $s[L], a[V/L], e[cEV^2/L^3]$



$$e[\theta] = \int_0^1 \left\{ \frac{1}{2} a^2(s)\theta_s^2 + \lambda cos\theta(s) \int_0^s a(t)dt \right\} ds,$$ где $\lambda = \frac{\rho g L^4}{cEV}$

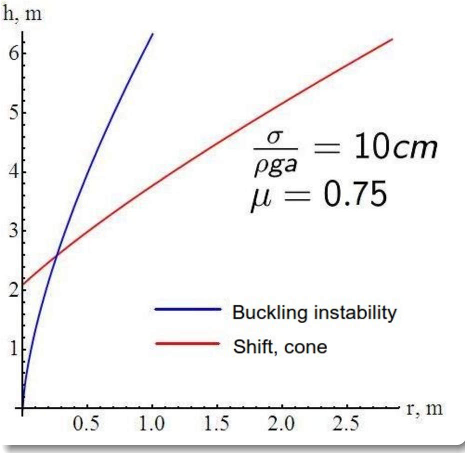
Buckling of arbitrary shape column under its own weight[5]

- Minimize energy: $\delta e/\delta\theta = 0$.
- $(a^2\theta_s)_s + \lambda sin\theta \int_0^s a(t)dt = 0$ - angular momentum balance eq-n obtained.
- $\theta(1) = 0, a^2\theta_s|_{s=0} = 0$ - bound. cond.
- small bend approx.: $sin\theta \approx \theta$, got eq-n on eigen values and eigen functions
- Maximize λ over possible column shapes: $\delta\lambda/\delta a = 0$. Obtain: $2(a\theta_s^2)_s + \lambda\theta^2 = 0$.
- System can be solved only numerically.
- **From cylindrical column close to destruction one can make column of arbitrary shape 1.47 times higher.**

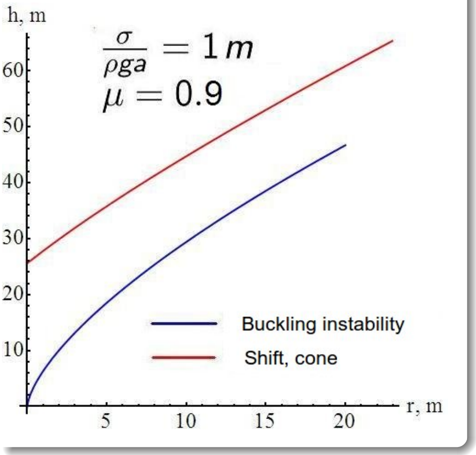


Two effects together

Typical sand from Wolga

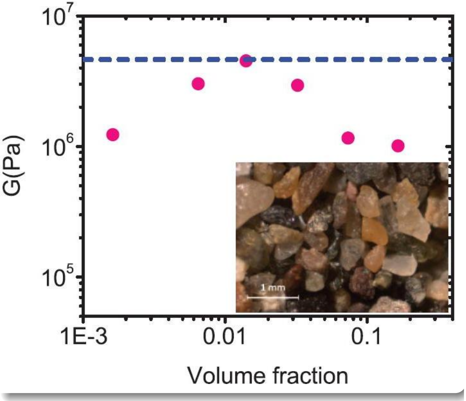


Hypothetical small-grained and strong sand

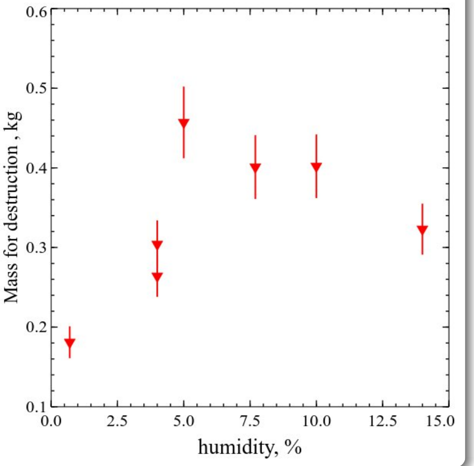


Optimal humidity

Optimization at low r: Young modulus[6,7]

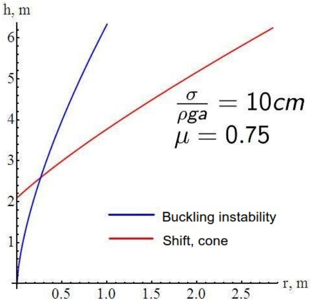


Optimization at high r: strength



Conclusions

- Sand is sticked together by water bridges.
- Two effects play role at destruction:
  - Buckling instability (for smaller areas)
  - Shift under own weight (for larger areas)
- From cylindrical column close to destruction one can make column of arbitrary shape 1.47 times higher.



Thank you for your attention!

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Literature

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150+ references to wet granular materials literature.

Young modulus of the wet sand

- Two spheres of radii R are pressed together. Herzian force F_{Herz} depends on δ фы $F_{Herz} = \frac{4\sqrt{2}}{9} R^{1/2} E_s \delta^{3/2}$.
- In case of water bridge between the spheres, eqiullibrium contraction δ_{eq} is defined from force balance: $2\pi\sigma R f(\phi) = \frac{4\sqrt{2}}{9} R^{1/2} E_s \delta_{eq}^{3/2}$.
- With external force exerted F_{ext}, one gets $\Delta\delta$: $F_{ext} = F_{Herz}(\delta_{eq} + \Delta\delta) - F_{Herz}(\delta_{eq}) \approx \frac{2\sqrt{2}}{3} R^{1/2} E_s \delta_{eq}^{1/2} \Delta\delta = \left(\frac{\pi}{24}\right)^{1/3} R^{2/3} E_s^{2/3} f(\phi)^{1/3} \sigma^{1/3} \Delta\delta = k\Delta\delta$.
- Arbitrary grains orientation: $\Delta\delta = \Delta l \cdot sin\theta$, $l = 2R \cdot sin\theta$, $F_{ext} = F \cdot sin\theta$, $E = \frac{F_{ext}/A}{\Delta l/l} = \frac{k}{2R} sin^3\theta$.
- Orientation averaging: $E = \frac{k}{4\pi R}$

Young modulus:

$$E = \frac{1}{8\pi} \left(\frac{\pi}{3}\right)^{1/3} \left(E_s^2 \sigma / R\right)^{1/3} f^{1/3}(\phi)$$

Experimentally checked at [7] for $\phi > 2 \cdot 10^{-3}\%$.