P01 : Cumulative Cannon

Problem statement: *How high may a ping-pong ball jump using the setup on the video? What is the maximal fraction of the total kinetic energy that can be transferred to the ball?*

INTRODUCTION

When a ping-pong ball is placed in a glass filled with water and released to fall from certain height to a flat floor, the ball is observed to shoot up to a substantially larger height. This phenomenon can be analogised with a popular demonstration of the classical Galilean cannon, where two balls of different elasticity are considered. With this motivation, we seek a theoretical model of this phenomenon and correlate the predictions from the experiments while identifying its' shortcomings.

This report tries to identify the relevant parameters which influences the phenomenon to answer these two questions

- What is the final height reached by the ping-pong ball given the relevant parameters known?
- What is the maximal fraction of total kinetic energy of the system that the ball receives?

We provide an effective model using relevant analogies identifying the primary parameters for the ball to shoot at the first place. Initially, we try using the Galilean cannon to treat the energy transfer from the water column in the vessel. To model the sudden shift of water pressure when the vessel hits the floor, we use the models studying water hammer effect. Certain empirical modifications in our model from the experiments are also proposed, alongside with future (or left) work.

THEORY

The Initial model of Galilean Cannon

There are two aspects of this phenomenon which can be largely understood. First, there is an effective transfer of energy to the ping-pong ball which is a consequence of an effective collision. We start by taking the model of Galilean cannon, which captures the basic principle of it very well.

The second aspect includes the behaviour of the fluid in the vessel when it makes a collision with the ground. A natural approximation to it's behaviour is to consider the column of the fluid to be momentarily rigid during the collision. As water is a incompressible fluid, we expect this behaviour in situations where the pressure over the column has a very sudden change. With this assumption, the expression of the velocity of the ball after the collision was found out to be

$$v_1 = \frac{v_0[\mu - e_2 - e_{12}(1+e_2)]}{(\mu+1)} \tag{1}$$

where,

- v_0 is the free falling velocity of the system
- μ is the ratio of mass of the ball to mass of the water column
- e_2 is the coefficient of the restitution between the vessel and ground
- e_{12} is the *effective* coefficient of restitution between the ball and water column.

It is evident here that e_{12} captures the second aspect of this process, which is unreasonable to calculate as a constant for the setup with water and ball. An important aspect which we neglected here is the relative motion of the fluid in the vessel (w.r.t floor) when it hits to the floor. Initially, the fluid column is assumed to have a constant velocity downward. As the collision occur, there is a sudden change in fluid momentum, experienced initially by the bottom-most section of the column. This leads to create a pressure pulse which propagates upward. A similar phenomenon in this case is studied in the pipe analysis of fluids, known as the water hammer effect.

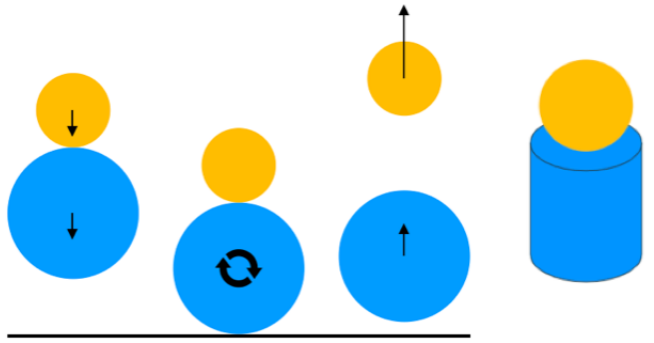

Fig. 1. Model of a Galilean cannon : (On Left) Two objects of different masses (and elastic properties) are allowed to collide on the ground. Here, a rigid ball with lighter mass is on the top of a larger elastic ball. Consequently, the lighter ball launches considerably higher than the height of the COM of the whole system. (On Right) Cartoon of the water column over which ping-pong ball reside.

Using the Water hammer effect

When the velocity of a fluid in a pipe changes after valve closure, there is a change in the fluid momentum. Water hammer effect is the special case where the velocity of water at one end in the water column is suddenly brought down to zero, which creates pressure transients in it.

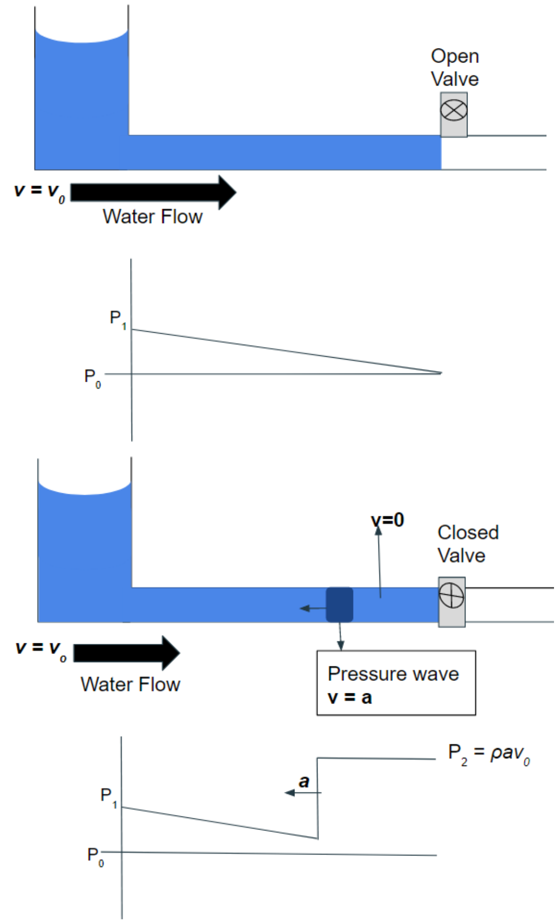

Fig. 2. Water hammer effect before and after valve closure : The pressure wave generated is used to model the collision

The magnitude of pressure difference for instantaneous valve closure created, as given by the Joukowsky equation, is

$$\Delta P = \rho C \Delta v \tag{2}$$

where,

- ΔP is change in pressure
- ρ is the density of the fluid
- C is the sonic velocity in the vessel
- Δv is the change in fluid velocity.

Calculating the sonic velocity in traditional sense involves various methods. The commonly referred equation used is

$$C = \sqrt{\frac{1}{\rho\left(\frac{1}{K} + \frac{D}{Ez}\right)}} \tag{3}$$

where

- K is bulk modulus of the fluid
- D is the diameter of the vessel
- E is Young modulus of the vessel/pipe
- z is the thickness of the vessel/pipe.

It is evident to understand that the material of the vessel (or a semi-closed pipe for that matter) influences the pressure difference.

We associate the water hammer effect to explain the shooting of the ping-pong ball with the pressure wave created. This pressure wave is the source of momentum transfer to the ball. The collision of vessel's base with the surface is similar to closing the valve instantaneously, where the magnitude of pressure pulse generated is given by (2). With the assumption that this process is plausible towards explaining the phenomenon, it is convenient to use the impulse theorem given to find the momentum transferred to the ball -

$$\Delta p = \int F dt. \tag{4}$$

General assumptions

These are the important assumptions that we make while modelling the system

1) The momentum transfer maximally occurs during the state of the ball when it is dipped inside the water column.
2) The velocity of the ball is higher than that of water splashing during collision
3) The water column is in 1 atm isobaric state during free fall i.e. there is no pressure gradient during while the vessel is falling.
4) The pressure impulse made in the vessel is aligned in vertical direction, for which effects of gravity is neglected
5) The ball and the vessel filled with fluid are taken to have the same acceleration

We have also introduced terminologies to account for the state of the ball in the vessel during the collision as follows

- Dip length (h) : It is the height of the spherical cap which is immersed in the liquid with respect to the lowest meniscus of the fluid surface
- Effective radius (r) : It is the maximal radial length of the immersed cross section of the ball.

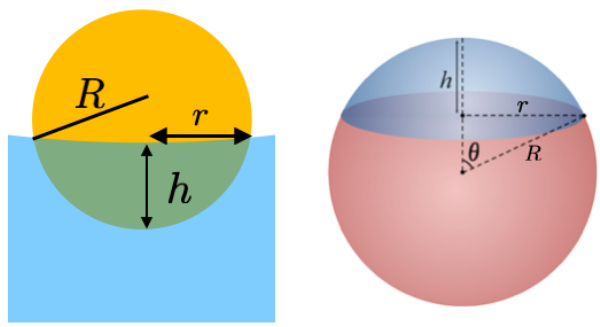

Fig. 3. (left) Cross section of the vessel system indicating dip length h, effective radius r and radius of the ball R. (right) The ping pong ball in 3D sphere, where θ is the cross section angle of the spherical cap

In a static system, the dip length can be calculated easily by [Derivation 1]

$$h^3 - 3Rh^2 + \frac{3m}{\rho_w \pi} = 0 \tag{5}$$

where

- R is the radius of the ball
- m is the mass of the ball
- ρ_w is the density of water

which is found by equating the buoyancy force exerted on the ball with it's weight.

Impact time and surface tension

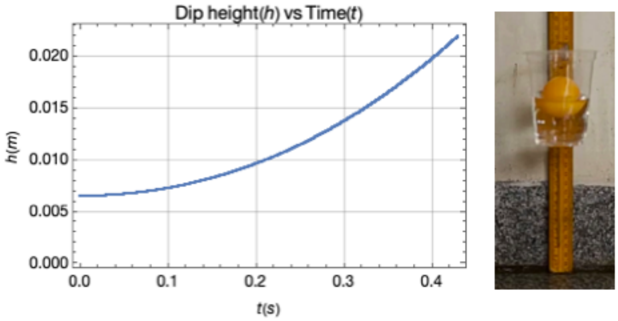

Fig. 4. (left) The change of dip height w.r.t time during the free fall with contact angle of $86.5deg$ [7], $R = 2cm$ and standard values for ρ_w, S generated from [9] (right) Dip of ball is close to 2 cm during free fall in a a 90cm vessel drop

From the first assumption, the time period when the momentum transfer to the ball takes place during the collision is

$$\Delta t = \frac{h}{C}. \tag{6}$$

The surface tension of the water layer plays a role in pulling the ball inside the vessel during free fall, which effectively increases the dip length. The surface force (F_{ff}) on the ball can be calculated as

$$F_{ff} = \underbrace{2\pi r S cos(\omega)}_{\text{Surface tension}} - \underbrace{\frac{1}{2}\rho_w A C_d \dot{h}^2}_{\text{Drag}} \tag{7}$$

where,

- $r = \sqrt{R^2 - (R-h)^2}, A = \pi r^2$
- S is the surface tension between the water and the ball
- R is the radius of the ball
- C_d is the drag coefficient of the sphere ($= 0.47$)
- ω is the contact angle of the water-ball interface.

The following force would be applicable on the ball until it fully immerses in the liquid. The effects are prominent when the vessel is dropped at larger heights which then needs to be considered for the correction of the dip height. From figure 4, we can consider these effects to be negligible for shorter fall distances.

Post Collision and height of the ball

By using (2) with Δv (change in velocity of fluid) to be the initial velocity of the vessel v_0 (as fluid comes to rest just after collision), surface area of contact $A = \pi r^2$ and

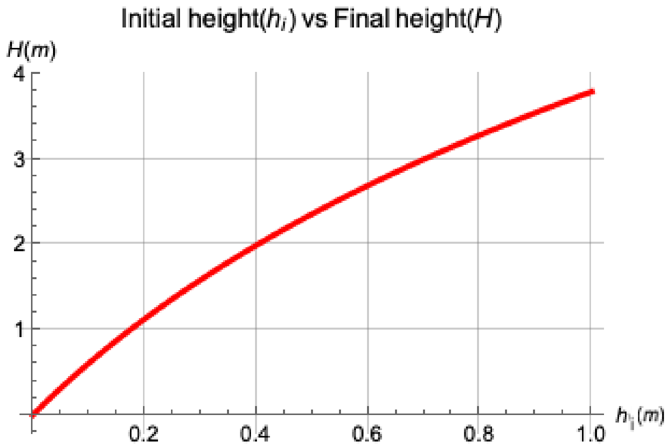

Fig. 5. Final height of the ball vs initial height of the vessel (in meters) for the dip length of 0.65 cm

time for momentum transfer (6), the momentum change of the ball is given by

$$m(v_f - v_0) = \rho_w C v_0 (\pi r^2)\frac{h}{C} \tag{8}$$

which allows us to calculate the final velocity of the ball v_f. Furthermore, the ball will experience drag as it shoots up (g is the gravitational acceleration), for which we use

$$m\dot{v} = -mg - \underbrace{\frac{\rho_{air}\pi R^2 C_d v^2}{2}}_{\text{drag}} \tag{9}$$

from which we calculate the maximum height (H) of the launched ball as [Derivation 2]

$$\boxed{H = \frac{m}{\rho_{air}\pi R^2 C_d}\ln\left(1 + \frac{\rho_{air}\pi R^2 C_d v_f^2}{2mg}\right).} \tag{10}$$

EXPERIMENT

Experimental setup

The experimental setup consisted of three metre scales (with least count of 1 cm) pasted along a wall in order to note the height, from which the ball-glass system is dropped and the final height reached by the ping-pong ball along a vertically straight line.

We have used standard ping-pong balls of mass (m) 2.7 gram and radius (R) 20 mm, and disposable plastic cups throughout the experiments.

Multiple cameras were fitted parallel to the scales at different heights to read initial and final height of ball. The setup was so installed in order to minimize parallax in observations. This led to the following experimental setup [8]. For calculating the initial dip of the ball, a ping-pong ball was placed in a glass filled with water and a vertical vernier caliper was lowered in the glass to read the dip. The experiment was repeated sufficiently to minimize the error in

Fig. 6. Experimental setup - three cameras to read the ball height and the base collision; vertical metre scale fixed along the wall

obtaining an accurate value of the initial dip closely relating to the theoretical one (0.65 cm, using (5)).

Results

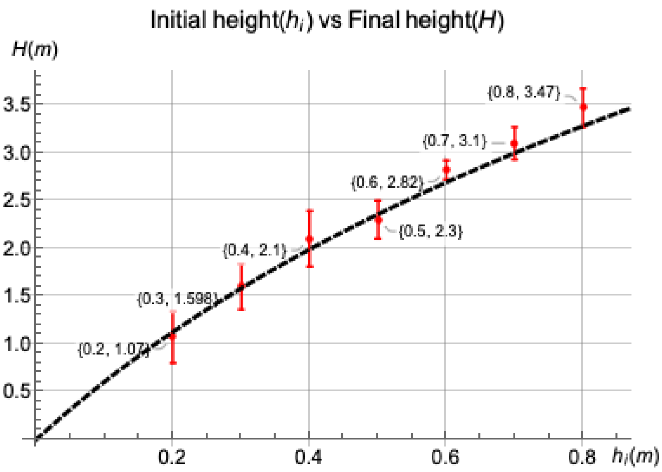

Fig. 7. Obtained results for the experiments with the theoretical predictions given in blue line

Plot 7 shows the experimental and theoretical data of the height reached by the ping pong ball by varying the height of the vessel fall. The error bars are for the height reached by the ball after releasing the setup from the specific heights taken. This show a fair agreement of our theory.

The maximal fraction of the total kinetic energy that can be transferred to the ball is calculated using the ratio of kinetic energy of the ball just after impact (using (8)) to the total energy (i.e. the potential energy of the system at the drop height), which is tabulated in Table 1.

The kinetic energy fraction is constant as obtained from the theoretical model. Obtaining the same in experiments requires a setup with a high speed camera to measure the ejection velocity.

TABLE I

TABULATION OF THEORETICAL FRACTION OF K.E. TRANSFERRED TO THE BALL

Initial he -ight (m)	Total K.E. (J)	K.E. just after im -pact (for ball) (J)	Fraction of K.E. transferred
0.20	0.158	0.036	0.23
0.30	0.238	0.054	0.23
0.40	0.317	0.073	0.23
0.50	0.397	0.092	0.23
0.60	0.476	0.109	0.23
0.70	0.556	0.128	0.23
0.80	0.635	0.146	0.23

Improving the model and future work

- We can add an ‘e’ factor in the model to take into account the energy losses in inelastic collision and the energy loss due to vibration of sides of cup. The ‘e’ factor will be a purely experimental factor similar to coefficient of restitution and has to be calculated for a given setup, the deviation observed in our data is due to the energy loss in inelastic collision and hence we can incorporate the ‘e’ factor to the equations as

$$v_f = v_0 \left[1 + e\rho(\pi r^2)\frac{d}{m}\right], \qquad (11)$$

  where ’e’ is the e factor to be calculated experimentally to find energy losses from the vibrations of the vessel.
- The deviations in the theoretical vs experimental graphs is also due to our considerations for the dip length to be constant. As seen in figure (4), the dip length (h) and effective radius (r) increases significantly for higher heights due to surface tension and leading to more momentum transfer. We can then alter the theoretical graph to fit for the higher drop distances by considering the variations in the d and r. Investigations are ongoing in those directions.

CONCLUSION

We identified the phenomenon of pressure wave in the water hammer effect to approximate the immediate pressure variation of the water column in the vessel during impact, which leads the ping-pong ball to shoot off. Then we have able to establish the relationship between the final height of the ball with the drop height of the vessel. With the lack of experimental precision, calculating the final speed of the ball was not possible for which the we presented a theoretical fraction of K.E transfer to be about 0.23.

From the conducted experiments, we are in the verge to investigate if the surface tension of the liquid has a role in the observed deviance of the result at higher drop heights from the theoretical results (as in figure 7). Further investigations towards the material of the vessel (in our case, a plastic disposable cup) can be replaced by using different variants of the available pipes.

FUTURE WORK

- As the water rises along the sides during free fall and the pressure wave is travelling upwards, upon collision the surface would converge towards center and is predicted to form a structure which can be represented, in an ideal case, by a Bessel's function of first kind. This is verified in [4] where the surface of coffee in a cup was analysed. We believe this has to be a reason for which different launching heights of the ping-pong ball will be obtained by using different vessels. We will study it further to find the optimal shape of container.

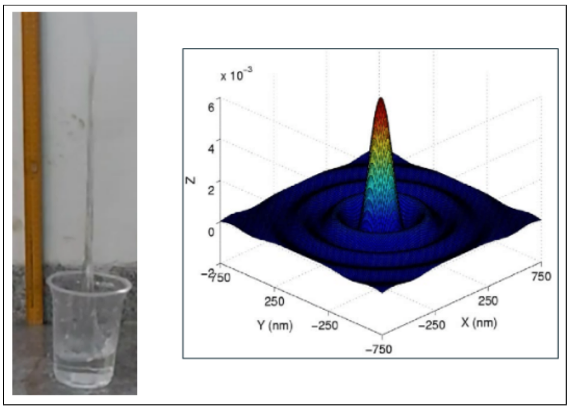

Fig. 8. Water column structure just after impact with surface and to the right is the Bessel's function plot (Google images)

- From our initial study with the provided video [5], it has been identified that the ping-pong ball does not immediately launch from the vessel upon collision. An associated time lag between the initial contact of the vessel and final launch of the ball has been noticed, which can offer a possible way to verify if the corrections to our dip length with increased water column height has significance. The delay interval can be observed easily by doing the experiment with a longer vessel. We intend to study it to verify our hypothesis regarding the dip length.

References

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[2] Water Hammer effect - https://www.fluidmechanics.co.uk/hydraulic-calculations/water-hammer-2/

[3] Joukowsky Equation in https://en.wikipedia.org/wiki/Water_hammer

[4] Water surface structure- ME201/MTH281/ME400/CHE400 Coffee Cup Oscillations, University of Rochester (http://www2.me.rochester.edu/courses/ME201/webexamp/coffee.pdf)

[5] https://youtu.be/mPOcFSHyd9o (Clip from 8:00 to 8:20)

[6] http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/tralon.html

[7] https://drive.google.com/file/d/1KHaE86bVgKVhR-hOYNoSnDstwoE-gKgm/view?usp=sharing

[8] https://youtu.be/0GIpplcXiZk

[9] https://drive.google.com/open?id=19x2jWRaBLlCLN09uApG9Wjf3-KT_PF_BK (Mathematica Code)

DERIVATIONS

Derivation 1:

$$mg = \rho_w V g$$

$$V = \pi \frac{d^2}{3}(3R - h)$$

$$m = \rho_w \pi \frac{d^2}{3}(3R - h)$$

$$h^3 - 3Rh^2 + \frac{3m}{\rho_w \pi} = 0$$

Derivation 2:

$$m\dot{v} = -mg - \frac{\rho A C_d v^2}{2}$$

$$v\frac{dv}{dx} = -\frac{\rho A C_d v^2}{2m} - g$$

$$dx = -\frac{vdv}{g + \frac{\rho A C_d v^2}{2m}}$$

let, $u = g + \dfrac{\rho A C_d v^2}{2m}$

$$du = \frac{\rho A C_d v dv}{m}$$

substituting from above,

$$\int dx = \frac{-m}{\rho A C_d} \int \frac{du}{u}$$

$$[x]_0^H = -\frac{m}{\rho A C_d}\left[\ln\left(g + \frac{\rho A C_d v_f^2}{2m}\right)\right]_{v_f}^{0}$$

$$\boxed{\text{Final height} = H = \frac{m}{\rho A C_d} \ln\left(\frac{g + \frac{\rho A C_d v_f^2}{2m}}{g}\right)}$$